ON COHEN-MACAULAY RINGS OF INVARIANTS

M. LORENZ AND J. PATHAK

Abstract. We investigate the transfer of the Cohen-Macaulay property from a commutative ring to a subring of invariants under the action of a finite group. Our point of view is ring theoretic and not a priori tailored to a particular type of group action. As an illustration, we briefly discuss the special case of multiplicative actions, that is, actions on group algebras $k[\mathbb{Z}^n]$ via an action on $\mathbb{Z}^n$.

Introduction

This article addresses the question to what extent the Cohen-Macaulay property passes from a (commutative) ring R to a subring R^G of invariants under the action of a finite group G on R. As is well-known, the Cohen-Macaulay property is indeed inherited by R^G whenever the trace map $\operatorname{tr}_G \colon R \to R^G$, $r \mapsto \sum_{g \in G} g(r)$, is surjective ([HE]; see also Section 3.2 below). In the opposite case, however, the property usually does not transfer, even in the particular case of linear actions, that is, G-actions on polynomial algebras $R = k[X_1, \ldots, X_n]$ by linear substitutions of the variables. The Cohen-Macaulay problem for linear invariants has been rather thoroughly explored without, at present, being anywhere near a final solution.

Our focus in this article will not be on linear G-actions on polynomial algebras nor, for the most part, on any other kind of group action on affine algebras over a field. Rather, in Sections 1 – 5, we work entirely in the setting of commutative noetherian rings. Besides being marginally more general, this approach has resulted in a number of simplifications of results previously obtained by Kemper [Ke$_1$], [Ke$_2$] in a geometric setting using geometric methods. Nevertheless, the article owes a great deal to Kemper's insights and originated from a study of his work.

A rough outline of the contents is as follows. Section 1 is devoted to relative trace maps. We determine the height of their image, an ideal of R^G, and use this result to give a lower bound for the height of annihilators in R^G of certain cohomology classes. Section 2 reviews basic material on Cohen-Macaulay rings

1991 *Mathematics Subject Classification.* 13A50, 16W22, 13C14, 13H10.

Key words and phrases. finite group action, ring of invariants, invariant theory, height, depth, Cohen-Macaulay ring, cohomology, Sylow subgroup.

Research of both authors supported in part by NSF grant DMS-9988756.

and local cohomology and describes a pair of spectral sequences constructed by Ellingsrud and Skjelbred [ES]. These are used to derive certain depth estimates. In Section 3, we return to rings of invariants R^G and note some easy facts on the non-Cohen-Macaulay locus of R^G and on the special case of Galois actions; it turns out that if the G-action on R is Galois in the sense of Auslander and Goldman [AG] then R^G is Cohen-Macaulay if and only if R is. Section 4 develops the main technical tools of this article. We use the aforementioned spectral sequences of Ellingsrud and Skjelbred to derive a depth formula for modules of invariants which underlies our subsequent applications. The latter concern the case where R has characteristic p and focus on the role played by the Sylow p-subgroup of G. For the precise statements of these results, we refer the reader to Section 5 where they are presented. The final Section 6 initiates the study of the Cohen-Macaulay property in the special case of multiplicative actions. These are defined to be G-actions on Laurent polynomial algebras $R = k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ stabilizing the lattice of monomials $\langle X_1, \dots, X_n \rangle \cong \mathbb{Z}^n$; so we may think of G as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. We show that if G maps onto some non-trivial p-group and has a cyclic Sylow p-subgroup, P, then R^G is Cohen-Macaulay if and only if P is generated by a bireflection, that is, a matrix $g \in \mathrm{GL}_n(\mathbb{Z})$ so that $g - 1_{n\times n}$ has rank at most 2. In this case, P must have order 2, 3, or 4. A more detailed study of the Cohen-Macaulay property for multiplicative invariants will form the subject of the second author's Ph.D. thesis.

Notations and Conventions. Throughout, G will denote a finite group and R will be a commutative ring on which G acts by ring automorphisms, $r \mapsto g(r)$. The subring of G-invariant elements of R will be denoted by R^G and the skew group ring of G over R by RG. Thus, RG is the free left R-module with basis the elements of G, made into a ring by means of the multiplication rule $rg \cdot r'g' = rg(r')gg'$ for $r, r' \in R$, $g, g' \in G$. The ring R is a module over RG via $rg \cdot r' = rg(r')$. All modules are understood to be left modules.

1. The relative trace map

1.1. Throughout this section, H denotes a subgroup of G. The *relative trace map* $\mathrm{tr}_{G/H} : R^H \to R^G$ is defined by

$$\mathrm{tr}_{G/H}(r) = \sum_{g \in G/H} g(r) \qquad (r \in R^H) \ .$$

Here, g runs over any transversal for the cosets gH of H in G. Since $\mathrm{tr}_{G/H}$ is R^G-linear, the image of $\mathrm{tr}_{G/H}$ is an ideal of R^G which we shall denote by

$$R^G_H \ .$$

1.2. **Covering primes.** The proof of the following lemma was communicated to us by Don Passman. The special case where R is an affine algebra over a field is covered by [Ke$_2$, Satz 4.7]. As usual, we will write ${}^gH = gHg^{-1}$ $(g \in G)$ and $I_G(\mathfrak{Q}) = \{g \in G \mid (g-1)(R) \subset \mathfrak{Q}\}$ denotes the *inertia group* of an ideal $\mathfrak{Q}$ of R.

Lemma 1.1. *For any prime ideal $\mathfrak{Q}$ of R,*

$$\mathfrak{Q} \supseteq R_H^G \iff [I_G(\mathfrak{Q}) : I_{{}^gH}(\mathfrak{Q})] \in \mathfrak{Q} \quad \textit{for all } g \in G$$

Proof. The implication $\Leftarrow$ follows from the straightforward formula

$$\mathrm{tr}_{G/H}(r) \equiv \sum_{g \in I_G(H) \backslash G/H} [I_G(\mathfrak{Q}) : I_{{}^gH}(\mathfrak{Q})]\, g(r) \mod \mathfrak{Q}$$

for all $r \in R^H$. For $\Rightarrow$, assume that $\mathfrak{Q} \supseteq R_H^G$. It suffices to show that

$$[I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \in \mathfrak{Q}\ .$$

Indeed, $R_H^G = R_{{}^gH}^G$, since $\mathrm{tr}_{G/H}(r) = \mathrm{tr}_{G/{}^gH}(g(r))$ holds for all $r \in R^H$ and $g \in G$.

To simplify notation, put $I = I_G(\mathfrak{Q})$ and let P denote a Sylow p-subgroup of $I \cap H = I_H(\mathfrak{Q})$, where $p \geq 0$ is the characteristic of the commutative domain $R/\mathfrak{Q}$. (Here $P = \{1\}$ if $p = 0$.) Then our desired conclusion, $[I : I \cap H] \in \mathfrak{Q}$, is equivalent with

$$[I : P] \in \mathfrak{Q}\ .$$

Furthermore, our assumption $\mathfrak{Q} \supseteq R_H^G$ entails that $\mathfrak{Q} \supseteq R_P^G$, because $\mathrm{tr}_{G/P} = \mathrm{tr}_{G/H} \circ \mathrm{tr}_{H/P}$. Thus, leaving H for P, we may assume that $H = P$ is a p-subgroup of I. Let $D = \{g \in G \mid g(\mathfrak{Q}) = \mathfrak{Q}\}$ denote the decomposition group of $\mathfrak{Q}$; so $I \leq D$. We claim that

$$\mathfrak{Q} \supseteq R_P^D\ .$$

To see this, choose $r \in R$ so that $r \in g(\mathfrak{Q})$ for all $g \in G \setminus D$ but $r \notin \mathfrak{Q}$. Then $s = \prod_{g \in D} g(r)$ also belongs to $\bigcap_{g \in G \setminus D} g(\mathfrak{Q})$ but not to $\mathfrak{Q}$ and, in addition, $s \in R^D$. Now assume that, contrary to our claim, there exists an element $f \in R^P$ so that $\mathrm{tr}_{D/P}(f) \notin \mathfrak{Q}$. Then $\mathrm{tr}_{D/P}(sf) = s\,\mathrm{tr}_{D/P}(f) \in \bigcap_{g \in G \setminus D} g(\mathfrak{Q}) \setminus \mathfrak{Q}$, and hence $\mathrm{tr}_{G/P}(sf) \notin \mathfrak{Q}$, a contradiction.

By the claim, we may replace G by D, thereby reducing to the case where $\mathfrak{Q}$ is G-stable. (Note that I is unaffected by this replacement.) So G acts on $R/\mathfrak{Q}$ with kernel I, P is a p-subgroup of I, and $R_P^G \subseteq \mathfrak{Q}$. Thus, $0 \equiv \mathrm{tr}_{G/P}(r) \equiv [I : P] \cdot \sum_{g \in G/I} g(r) \mod \mathfrak{Q}$ holds for all $r \in R^P$. Our desired conclusion, $[I : P] \in \mathfrak{Q}$, will follow if we can show that $\sum_{g \in G/I} g(r) \notin \mathfrak{Q}$ holds for some $r \in R^P$. But $\sum_{g \in G/I} g$ induces a nonzero endomorphism on $R/\mathfrak{Q}$, by linear independence of automorphisms of $K = \mathrm{Fract}(R/\mathfrak{Q})$; so $\sum_{g \in G/I} g(s) \notin \mathfrak{Q}$ holds for some $s \in R$. Putting $r = \prod_{h \in P} h(s)$, we have $r \in R^P$ and $r \equiv s^{|P|}$

mod $\mathfrak{Q}$. Since $|P|$ is 1 or a power of $p = \operatorname{char} K$, we obtain $\sum_{g\in G/I} g(r) \equiv \sum_{g\in G/I} g(s^{|P|}) \equiv \left(\sum_{g\in G/I} g(s)\right)^{|P|} \notin \mathfrak{Q}$, as required. $\square$

1.3. **Height formula.** For any collection $\mathcal{X}$ of subgroups of G, we define the ideal $R^G_{\mathcal{X}}$ of R^G by

$$R^G_{\mathcal{X}} = \sum_{H\in\mathcal{X}} R^G_H \ .$$

Inasmuch as $R^G_D \subseteq R^G_H = R^G_{{}^gH}$ holds for all $D \le H \le G$ and $g \in G$, there is no loss in assuming that $\mathcal{X}$ is closed under G-conjugation and under taking subgroups.

Moreover, for any subgroup $H \le G$, we define

$$I_R(H) = \sum_{h\in H} (h-1)(R)R \ .$$

Thus, $I_R(H)$ is an ideal of R, and $\mathfrak{Q} \supseteq I_R(H)$ is equivalent with $H \le I_G(\mathfrak{Q})$.

Lemma 1.2. *Assume that $\mathbb{F}_p \subseteq R$, and let $\mathcal{X}$ be a collection of subgroups of G that is closed under G-conjugation and under taking subgroups. Then*

$$\operatorname{height} R^G_{\mathcal{X}} = \inf\{\operatorname{height} I_R(P) \mid P \text{ is a } p\text{-subgroup of } G,\ P \notin \mathcal{X}\} \ .$$

Proof. One has

$$\operatorname{height} R^G_{\mathcal{X}} = \inf_{\mathfrak{q}} \operatorname{height} \mathfrak{q} = \inf_{\mathfrak{Q}} \operatorname{height} \mathfrak{Q} \ ,$$

where $\mathfrak{q}$ runs over the prime ideals of R^G containing $R^G_{\mathcal{X}}$ and $\mathfrak{Q}$ runs over the primes of R containing $R^G_{\mathcal{X}}$. Here, the first equality is just the definition of height, while the second equality is a consequence of the standard relations between the primes of R and R^G; see, e.g., [Bou, Théorème 2 on p. 42].

By Lemma 1.1,

$$\mathfrak{Q} \supseteq R^G_{\mathcal{X}} \iff p \,\big|\, [I_G(\mathfrak{Q}) : I_H(\mathfrak{Q})] \text{ for all } H \in \mathcal{X}.$$

Since $I_H(\mathfrak{Q}) = I_G(\mathfrak{Q}) \cap H$ belongs to $\mathcal{X}$ for $H \in \mathcal{X}$, the latter condition just says that the Sylow p-subgroups of $I_G(\mathfrak{Q})$ do not belong to $\mathcal{X}$ or, equivalently, some p-subgroup $P \le I_G(\mathfrak{Q})$ does not belong to $\mathcal{X}$. Therefore,

$$\mathfrak{Q} \supseteq R^G_{\mathcal{X}} \iff \mathfrak{Q} \supseteq \bigcap_{P \le G \text{ a } p\text{-subgroup},\ P \notin \mathcal{X}} I_R(P) \ ,$$

which implies the asserted height formula. $\square$

1.4. **Annihilators of cohomology classes.** Let M be a module over the skew group ring RG. Then, for each $r \in R^G$, the map $\rho\colon M \to M$, $m \mapsto rm$, is G-equivariant, and hence ρ induces a map on cohomology $\rho_* : H^*(G,M) \to H^*(G,M)$. Letting r act on $H^*(G,M)$ via ρ_* we make $H^*(G,M)$ into an R^G-module.

Lemma 1.3. *The ideal R^G_H of R^G annihilates the kernel of the restriction map $\operatorname{res}^G_H : H^*(G,M) \to H^*(H,M)$.*

Proof. The action of $R^G = H^0(G,R)$ on $H^*(G,M)$ can also be interpreted as coming from the cup product

$$H^0(G,R) \times H^*(G,M) \xrightarrow{\cup} H^*(G,R\otimes M) \xrightarrow{\cdot} H^*(G,M) \ ,$$

where the map denoted by $\cdot$ comes from the G-equivariant map $R \otimes M \to M$, $r \otimes m \mapsto rm$; see, e.g., [Br, Exerc. 1 on p. 114]. Furthermore, the relative trace map $\operatorname{tr}_{G/H} : R^H \to R^G$ is identical with the corestriction map $\operatorname{cor}^G_H : H^0(H,R) \to H^0(G,R)$; cf. [Br, p. 81]. Thus, the transfer formula for cup products ([Br, (3.8) on p. 112]) gives, for $s \in R^H$ and $x \in H^*(G,M)$,

$$\operatorname{tr}_{G/H}(s)x = \cdot(\operatorname{tr}_{G/H}(s) \cup x) = \cdot(\operatorname{cor}^G_H(s \cup \operatorname{res}^G_H(x))) \ .$$

Therefore, if $\operatorname{res}^G_H(x) = 0$ then $\operatorname{tr}_{G/H}(s)x = 0$. □

We summarize the material of this section in the following proposition. For convenience, we write $\operatorname{res}^G_P(\,.\,) = \,.\,\big|_P$.

Proposition 1.4. *Assume that $\mathbb{F}_p \subseteq R$, and let M be an RG-module. Then, for any $x \in H^*(G,M)$,*

$$\operatorname{height} \operatorname{ann}_{R^G}(x) \geq \inf\{\operatorname{height} I_R(P) \mid P \textit{ a } p\textit{-subgroup of } G,\ x\big|_P \neq 0\} \ .$$

Proof. Let $\mathcal{X}$ denote the splitting data of x, that is, $\mathcal{X} = \{H \leq G \mid x\big|_H = 0\}$. By Lemma 1.3, $\operatorname{ann}_{R^G}(x) \supseteq R^G_{\mathcal{X}}$, and by Lemma 1.2, $\operatorname{height} R^G_{\mathcal{X}} = \inf\{\operatorname{height} I_R(P) \mid P$ is a p-subgroup of G, $x\big|_P \neq 0\}$. The proposition follows. □

2. Depth

2.1. In this section, A denotes any commutative noetherian ring, $\mathfrak{a}$ is an ideal of A, and M denotes a finitely generated module over the group ring $A[G]$.

2.2. **Depth and local cohomology.** Let $H^i_{\mathfrak{a}}$ denote the i-th local cohomology functor with respect to $\mathfrak{a}$, that is, the i-th right derived functor of the $\mathfrak{a}$-torsion functor

$$\Gamma_{\mathfrak{a}}(M) = H^0_{\mathfrak{a}}(M) = \{m \in M \mid m \text{ is annihilated by some power of } \mathfrak{a}\} \ .$$

Then

$$\operatorname{depth}(\mathfrak{a}, M) = \inf\{i \mid H^i_{\mathfrak{a}}(M) \neq 0\}$$

(where $\inf \emptyset = \infty$); see [BS, Theorem 6.2.7].

Recall from Section 1.4 (with $A = R^G$) that $H^*(G, M)$ is a module over A. Our hypotheses on A and M entail that M is a noetherian A-module, and hence so are all $H^q(G, M)$. Therefore,

$$\operatorname{depth}(\mathfrak{a}, M) = \inf\{i \mid H^i_{\mathfrak{a}}(M) \neq 0\}$$

and

$$\operatorname{depth}(\mathfrak{a}, H^q(G, M)) = \inf\{i \mid H^i_{\mathfrak{a}}(H^q(G, M)) \neq 0\}\ .$$

All $H^i_{\mathfrak{a}}(M)$ are $A[G]$-modules, via the action of $A[G]$ on M.

2.3. **The Ellingsrud-Skjelbred spectral sequences.** The above A-modules $H^p_{\mathfrak{a}}(H^q(G, M))$ feature as the E_2^{pq}-terms of a certain spectral sequence due to Ellingsrud and Skelbred [ES]. In fact, two related spectral sequences are constructed in [ES] in the following manner.

The $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ and the G-fixed point functor $(\,.\,)^G = H^0(G, \,.\,)$ clearly commute: $\Gamma_{\mathfrak{a}}(M^G) = (\Gamma_{\mathfrak{a}}(M))^G$. Moreover, if the $A[G]$-module M is injective, then one checks that $\Gamma_{\mathfrak{a}}(M)$ is also injective as $A[G]$-module (as in [BS, Prop. 2.1.4]) and M^G is injective as A-module. Therefore, $H^i(G, \Gamma_{\mathfrak{a}}(M)) = 0$ and $H^i_{\mathfrak{a}}(M^G) = 0$ holds for all $i > 0$ if M is injective. We obtain two Grothendieck spectral sequences converging to $H^*_{\mathfrak{a}}(G, M) := R^*(\Gamma_{\mathfrak{a}}(\,.\,)^G)(M) = R^*((\,.\,)^G\Gamma_{\mathfrak{a}})(M)$, for any $A[G]$-module M; e.g., [Ro, Theorem 11.38]:

$$\begin{array}{ccc} E_2^{p,q} = H^p_{\mathfrak{a}}(H^q(G, M)) & & \\ & \searrow\!\!\searrow & \\ & & H^{p+q}_{\mathfrak{a}}(G, M) \\ & \nearrow\!\!\nearrow & \\ \mathcal{E}_2^{p,q} = H^p(G, H^q_{\mathfrak{a}}(M)) & & \end{array} \tag{2.1}$$

2.4. **Depth estimates.** The depth formulas in Section 2.2 combined with the spectral sequences (2.1) yield the following estimates for $\operatorname{depth}(\mathfrak{a}, M^G)$.

Lemma 2.1. (a) **lower bound:** $\operatorname{depth}(\mathfrak{a}, M^G) \geq \min\{\operatorname{depth}(\mathfrak{a}, M), h_{\mathfrak{a}} + 1\}$, *where* $h_{\mathfrak{a}} = \inf_{q>0}\{q + \operatorname{depth}(\mathfrak{a}, H^q(G, M))\}$.

(b) **upper bound:** *Assume that* $H^{p_0}_{\mathfrak{a}}(H^{q_0}(G, M)) \neq 0$ *for some* $p_0 \geq 0$, $q_0 > 0$ *with* $s = p_0 + q_0 < \operatorname{depth}(\mathfrak{a}, M)$. *Assume further that* $H^{s+1-\ell}_{\mathfrak{a}}(H^{\ell}(G, M)) = 0$ *holds for* $\ell = 1, \ldots, q_0 - 1$ *and* $H^{s-1-\ell}_{\mathfrak{a}}(H^{\ell}(G, M)) = 0$ *holds for* $\ell > q_0$. *Then* $\operatorname{depth}(\mathfrak{a}, M^G) \leq s + 1$.

Proof. Put $m = \operatorname{depth}(\mathfrak{a}, M)$. Then $H^q_{\mathfrak{a}}(M) = 0$ for $q < m$, and so the $\mathcal{E}$-sequence in (2.1) implies that $H^n_{\mathfrak{a}}(G, M) = 0$ for $n < m$. Therefore, the E-sequence satisfies

$$E^{p,q}_{\infty} = 0 \quad \text{if } p + q < m. \tag{2.2}$$

Furthermore, $E^{p,0}_2 = H^p_{\mathfrak{a}}(M^G)$; so

$$\operatorname{depth}(\mathfrak{a}, M^G) = \inf\{p \mid E^{p,0}_2 \neq 0\} \ .$$

Finally,

$$h_{\mathfrak{a}} = \inf\{p + q \mid q > 0, E^{p,q}_2 \neq 0\} \ .$$

To prove (a), assume that $p < \min\{m, h_{\mathfrak{a}} + 1\}$. Then $E^{p,0}_{\infty} = 0$, by (2.2), and $E^{i,j}_r = 0$ for $j > 0$, $i + j < p$, $r \geq 2$. Recall that the differential d_r of E_r has bidegree $(r, 1 - r)$. Thus, $E^{p,0}_r$ has no nontrivial boundaries and consists entirely of cycles. This shows that $E^{p,0}_2 = E^{p,0}_3 = \cdots = E^{p,0}_{\infty}$, and hence $E^{p,0}_2 = 0$. Thus, (a) is proved.

For (b), we check that $E^{s+1,0}_2 \neq 0$. Our hypotheses imply that, at position (p_0, q_0), all incoming differentials d_r $(r \geq 2)$ are 0 as well as all outgoing d_r $(r \geq 2, r \neq q_0 + 1)$. Therefore, $E^{p_0,q_0}_{q_0+1} = E^{p_0,q_0}_2$ and $E^{p_0,q_0}_{\infty} = E^{p_0,q_0}_{q_0+2} = \operatorname{Ker}(d^{p_0,q_0}_{q_0+1})$. The former implies that $E^{p_0,q_0}_{q_0+1} \neq 0$, by hypothesis in (p_0, q_0), and the latter shows that $d^{p_0,q_0}_{q_0+1}$ is injective, because $E^{p_0,q_0}_{\infty} = 0$ by (2.2). Thus, $d^{p_0,q_0}_{q_0+1}$ embeds $E^{p_0,q_0}_{q_0+1}$ into $E^{s+1,0}_{q_0+1}$, forcing the latter to be nonzero. Hence, $E^{s+1,0}_2$ is nonzero as well, as desired. ◻

2.5. **Cohen-Macaulay rings.** For any finitely generated A-module V, one defines $\dim V = \dim(A/\operatorname{ann}_A V)$ and

$$\operatorname{height}(\mathfrak{a}, V) = \operatorname{height}(\mathfrak{a} + \operatorname{ann}_A V/\operatorname{ann}_A V) \ ;$$

so $\dim V = \sup_{\mathfrak{a}} \operatorname{height}(\mathfrak{a}, V)$. Always,

$$\operatorname{depth}(\mathfrak{a}, V) \leq \operatorname{height}(\mathfrak{a}, V) \ ;$$

see [BH, Exerc. 1.2.22(a)]. The A-module V is called *Cohen-Macaulay* if equality holds for all ideals $\mathfrak{a}$ of A. In order to show that V is Cohen-Macaulay, it suffices to check that $\operatorname{depth}(\mathfrak{a}, V) \geq \operatorname{height}(\mathfrak{a}, V)$ holds for all maximal ideals $\mathfrak{a}$ of A with $\mathfrak{a} \supseteq \operatorname{ann}_A V$.

3. The Cohen-Macaulay property for invariant rings

3.1. We now return to invariant rings R^G. Our main objective is to investigate when the Cohen-Macaulay property passes from R to R^G. In this section, we record a few elementary observations that are independent of the local cohomology methods in Section 2.

3.2. **The non-Cohen-Macaulay locus.** By definition, the non-Cohen-Macaulay locus of R^G consists of those prime ideals $\mathfrak{q}$ of R^G so that the localization $(R^G)_\mathfrak{q}$ is not Cohen-Macaulay. Thus, R^G is Cohen-Macaulay if and only if its non-Cohen-Macaulay locus is empty. Here, we point out a general bound for the non-Cohen-Macaulay locus in terms of relative trace maps. More detailed results for affine algebras over a field can be found in [Ke$_2$, Kapitel 5]. Recall the notation $R^G_\mathcal{X}$ from Section 1.3.

Proposition 3.1. *Let $\mathcal{CM}$ denote the set of subgroups H of G so that R^H is Cohen-Macaulay. Then, for every prime ideal $\mathfrak{q}$ of R^G so that $\mathfrak{q} \not\supseteq R^G_{\mathcal{CM}}$, the localization $(R^G)_\mathfrak{q}$ is Cohen-Macaulay.*

Proof. By hypothesis, $\mathfrak{q} \not\supseteq R^G_H$ for some $H \in \mathcal{CM}$. Let $R_\mathfrak{q}$ denote the localization of R at the multiplicative subset $R^G \setminus \mathfrak{q}$. Then the G-action on R extends to $R_\mathfrak{q}$ and $(R_\mathfrak{q})^G = (R^G)_\mathfrak{q}$; see [Bou, Prop. 23 on p. 34]. Similarly, $(R_\mathfrak{q})^H = (R^H)_\mathfrak{q}$; so $(R_\mathfrak{q})^H$ is Cohen-Macaulay. By choice of $\mathfrak{q}$ the relative trace map $\operatorname{tr}_{G/H} \colon (R_\mathfrak{q})^H \to (R_\mathfrak{q})^G$ is onto. Fix an element $c \in (R_\mathfrak{q})^H$ so that $\operatorname{tr}_{G/H}(c) = 1$ and define $\rho \colon (R_\mathfrak{q})^H \to (R_\mathfrak{q})^G$ by $\rho(x) = \operatorname{tr}_{G/H}(cx)$. This map is a “Reynolds operator”, i.e., ρ is $(R_\mathfrak{q})^G$-linear and restricts to the identity on $(R_\mathfrak{q})^G$. Since $(R_\mathfrak{q})^H$ is integral over $(R_\mathfrak{q})^G$, a result of Hochster and Eagon ([HE] or [BH, Theorem 6.4.5]) implies that $(R_\mathfrak{q})^G$ is Cohen-Macaulay, which proves the proposition. $\square$

As an application, we note that if G has subgroups H_i so that each R^{H_i} is Cohen-Macaulay and the indices $[G : H_i]$ are coprime in R^G then R^G is Cohen-Macaulay as well. Indeed, writing $1 = \sum_i [G : H_i] r_i$ with $r_i \in R^G$, we obtain $1 = \sum_i \operatorname{tr}_{G/H_i}(r_i) \in R^G_{\mathcal{CM}}$; so the non-Cohen-Macaulay locus of R^G is empty.

3.3. **Galois actions.** Recall that the G-action on R is *Galois*, in the sense of Auslander and Goldman [AG], if every maximal ideal of R has trivial inertia group in G.

Proposition 3.2. *If the G-action on R is Galois then R^G is Cohen-Macaulay if and only if R is.*

Proof. By [CHR, Lemma 1.6 and Theorem 1.3], the trace map $\operatorname{tr}_{G/1} \colon R \to R^G$ is surjective for Galois actions and R is finitely generated projective as R^G-module. Thus, R is faithfully flat as R^G-module. Moreover, for any prime $\mathfrak{Q}$ of R and $\mathfrak{q} = \mathfrak{Q} \cap R^G$, the fibre $R_\mathfrak{Q}/\mathfrak{q}R_\mathfrak{Q}$ has dimension 0. Therefore, by [BH, 2.1.23], R^G is Cohen-Macaulay if and only if R is. $\square$

4. Modules of invariants

4.1. Throughout this section, R^G is assumed noetherian and $\mathfrak{a}$ denotes an ideal of R^G. Moreover, M denotes an RG-module that is finitely generated as R^G-module. Our finiteness assumptions hold, for example, whenever R is an affine algebra over some noetherian subring $k \subseteq R^G$ and M is a finitely generated RG-module; see [Bou, Théorème 2 on p. 33].

4.2. **The problem and a sufficient condition.** Assuming ${}_RM$ to be Cohen-Macaulay, we are interested in the question under what circumstances ${}_{R^G}M^G$ will be Cohen-Macaulay as well. We remark that ${}_RM$ is Cohen-Macaulay if and only if ${}_{R^G}M$ is; see [Ke_2, Proposition 1.17].

For future reference, we note the following simple lemma.

Lemma 4.1. *Assume that ${}_RM$ is Cohen-Macaulay and that $\sqrt{\mathfrak{a}} \supseteq \operatorname{ann}_{R^G} M^G$. Then* $\operatorname{depth}(\mathfrak{a}, M) = \operatorname{height}(\mathfrak{a}, M) \geq \operatorname{height}(\mathfrak{a}, M^G)$.

Proof. Note that $\sqrt{\mathfrak{a}} \supseteq \operatorname{ann}_{R^G} M^G \supseteq \operatorname{ann}_{R^G} M$ entails that $\operatorname{height}(\mathfrak{a}, M) \geq \operatorname{height}(\mathfrak{a}, M^G)$. Further, $\operatorname{height}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{a}, M)$, because ${}_{R^G}M$ is Cohen-Macaulay. The lemma follows. □

We now give a sufficient condition for ${}_{R^G}M^G$ to be Cohen-Macaulay. We note that $\dim {}_RM = \dim {}_{R^G}M$, by the usual relations between the primes of R and of R^G.

Corollary 4.2. *Assume that ${}_RM$ is Cohen-Macaulay. If $H^q(G, M) = 0$ holds for $0 < q < \dim {}_RM - 1$ then ${}_{R^G}M^G$ is Cohen-Macaulay as well.*

Proof. Let $\mathfrak{a}$ be an ideal of R^G with $\mathfrak{a} \supseteq \operatorname{ann}_{R^G} M^G$. Our hypothesis on $H^q(G, M)$ entails that the value of $h_{\mathfrak{a}}$ in Lemma 2.1 satisfies $h_{\mathfrak{a}} \geq \dim {}_RM - 1$. Also, $\dim_R M = \dim {}_{R^G}M \geq \operatorname{height}(\mathfrak{a}, M) \geq \operatorname{height}(\mathfrak{a}, M^G)$, by Lemma 4.1. Thus, Lemma 2.1(a) gives $\operatorname{depth}(\mathfrak{a}, M^G) \geq \operatorname{height}(\mathfrak{a}, M^G)$, as required. □

4.3. **Depth formula.** In view of Corollary 4.2, we may concentrate on the case where M has non-vanishing positive G-cohomology. The following proposition is a version of results of Kemper; see [Ke_1, Corollary 1.6] and [Ke_2, Kor. 1.18].

Proposition 4.3. *Assume that ${}_RM$ is Cohen-Macaulay and that $\sqrt{\mathfrak{a}} \supseteq \operatorname{ann}_{R^G} M^G$. Furthermore, assume that, for some $r \geq 0$, $H^q(G, M) = 0$ holds for $0 < q < r$ but $\mathfrak{a}x = 0$ for some $0 \neq x \in H^r(G, M)$. Then*

$$\operatorname{depth}(\mathfrak{a}, M^G) = \min\{r + 1, \operatorname{depth}(\mathfrak{a}, M)\} .$$

Remark. $\operatorname{height}(\mathfrak{a}, M) = \operatorname{depth}(\mathfrak{a}, M)$ holds in the above formula; see Lemma 4.1.

Proof of Proposition 4.3. Our hypothesis $\mathfrak{a}x = 0$ for some $0 \neq x \in H^r(G, M)$ is equivalent with $H^0_{\mathfrak{a}}(H^r(G, M)) \neq 0$; so $\operatorname{depth}(\mathfrak{a}, H^r(G, M)) = 0$. The asserted equality is trivial for $r = 0$, since $\operatorname{depth}(\mathfrak{a}, M^G) = \operatorname{depth}(\mathfrak{a}, M) = 0$ holds in this

case. Thus we assume that $r > 0$. Then, in the notation of Lemma 2.1, we have $r = h_{\mathfrak{a}}$, and part (a) of the lemma gives the inequality $\geq$.

To prove the reverse inequality, note that Lemma 4.1 gives $\operatorname{depth}(\mathfrak{a}, M) \geq \operatorname{depth}(\mathfrak{a}, M^G)$. Therefore, it suffices to show that $\operatorname{depth}(\mathfrak{a}, M^G) \leq r+1$ if $\operatorname{depth}(\mathfrak{a}, M) > r+1$. For this, we quote Lemma 2.1(b) with $p_0 = 0$ and $q_0 = r$ (so $s = r$). □

5. The Sylow subgroup of G

5.1. In this section, R is assumed to be noetherian as R^G-module. We further assume that $\mathbb{F}_p \subseteq R$ and we let P denote a Sylow p-subgroup of G.

5.2. **A necessary condition.** Put

$$\mu = \mu(G, R) = \inf\{r > 0 \mid H^r(G, R) \neq 0\} .$$

Proposition 5.1. *Put $\mathcal{P} = \{P' \leq P \mid \operatorname{height} I_R(P') \leq \mu + 1\}$. If R and R^G are both Cohen-Macaulay and $\mu < \infty$ then the restriction map*

$$\operatorname{res}^G_{\mathcal{P}} \colon H^\mu(G, R) \to \prod_{P' \in \mathcal{P}} H^\mu(P', R)$$

is injective.

Proof. Let $0 \neq x \in H^\mu(G, R)$ be given and put $\mathfrak{a} = \operatorname{ann}_{R^G}(x)$. Then, by Proposition 1.4,

$$\operatorname{height} \mathfrak{a} \geq \inf\{\operatorname{height} I_R(P') \mid P' \text{ a } p\text{-subgroup of } G,\ x\big|_{P'} \neq 0\} .$$

Since R^G is Cohen-Macaulay, $\operatorname{height} \mathfrak{a} = \operatorname{depth} \mathfrak{a}$. Finally, Proposition 4.3 with $M = R$ gives $\operatorname{depth} \mathfrak{a} \leq \mu + 1$. Thus, there exists a p-subgroup P' of G with $x\big|_{P'} \neq 0$ and $\operatorname{height} I_R(P') \leq \mu + 1$. Note that both the condition $x\big|_{P'} \neq 0$ and the value of $\operatorname{height} I_R(P')$ are preserved upon replacing P' by a conjugate ${}^gP'$ with $g \in G$. Therefore, we may assume that $P' \in \mathcal{P}$, which proves the proposition. □

5.3. **Fixed-point-free actions.** A subgroup H of G is said to act *fixed-point-freely* on R if $\operatorname{height} I_R(H') \geq \dim R$ holds for all $1 \neq H' \leq H$.

Corollary 5.2. *Assume that R is Cohen-Macaulay and that the Sylow p-subgroup of G acts fixed-point-freely on R. Then: R^G is Cohen-Macaulay if and only if* $\dim R \leq \mu + 1$.

Proof. The implication $\Leftarrow$ follows from Corollary 4.2 with $M = R$. For the converse, let R^G be Cohen-Macaulay and assume, without loss, that $\mu < \infty$. Then Proposition 5.1 implies that there is a subgroup $1 \neq P' \leq P$ with $\operatorname{height} I_R(P') \leq \mu + 1$. On the other hand, by hypothesis on the G_p-action, $\operatorname{height} I_R(P') \geq \dim R$; so $\dim R \leq \mu + 1$. □

5.4. **Bireflections.** Following [Ke_2], we will call an element $g \in G$ a *bireflection* on R if height $I_R(\langle g \rangle) \leq 2$.

Corollary 5.3. *Assume that R and R^G are Cohen-Macaulay. Let H denote the subgroup of G that is generated by all p'-elements of G and all bireflections in P. Then $R^G = R^G_H$.*

Proof. First note that H is a normal subgroup of G and G/H is a p-group. Thus, if $R^G \neq R^G_H$ or, equivalently, $\widehat{H}^0(G/H, R^H) \neq 0$ then also $H^1(G/H, R^H) \neq 0$; see [Br, Theorem VI.8.5]. In view of the exact sequence

$$0 \to H^1(G/H, R^H) \longrightarrow H^1(G, R) \xrightarrow{\operatorname{res}^G_H} H^1(H, R)$$

(see [Ba, 35.3]) we further obtain $H^1(G, R) \neq 0$. Thus, $\mu = 1$ holds in Proposition 5.1 and every $P' \in \mathcal{P}$ consists of bireflections. Therefore, $P' \subseteq H$ and Proposition 5.1 implies that $\operatorname{res}^G_H\colon H^1(G, R) \to H^1(H, R)$ is injective, contradicting the above exact sequence. Therefore, we must have $R^G = R^G_H$. $\square$

We remark that if $\mathbb{F}_p$ is a G-module direct summand of R then the equality $R^G = R^G_H$ forces $G = H$.

5.5. **The case $|P| = p$.** Put

$$\mu_p(G) = \mu(G, \mathbb{F}_p) = \inf\{r > 0 \mid H^r(G, \mathbb{F}_p) \neq 0\} \ .$$

We will determine this number in the case where the order of G is divisible by p but not by p^2; in other words, $|P| = p$. As usual $\mathbb{N}_G(P)$ and $\mathbb{C}_G(P)$ will denote the normalizer and the centralizer, respectively, of P in G. Thus, $\mathbb{N}_G(P)/\mathbb{C}_G(P)$ is a subgroup of $\operatorname{Aut}(P) = \operatorname{Aut}(\mathbb{Z}/p) \cong \mathbb{F}_p^*$, and hence it is cyclic of order dividing $p - 1$.

Corollary 5.4. *Assume that $|P| = p$. Then $\mu_p(G) = 2[\mathbb{N}_G(P) : \mathbb{C}_G(P)] - 1$. Moreover, if $\mathbb{F}_p$ is a G-module direct summand of R and R and R^G are both Cohen-Macaulay then* height $I_R(P) \leq 2[\mathbb{N}_G(P) : \mathbb{C}_G(P)]$.

Proof. Put $N = \mathbb{N}_G(P)$, $C = \mathbb{C}_G(P)$, and $r = 2[N : C] - 1$. In order to prove that $\mu_p(G) = r$, we use the fact that $H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)^{N/C}$ holds for $* > 0$; see [Be, Corollary 3.6.19]. If $p = 2$ then $N = C$ and so $r = 1$. Moreover, $H^*(P, \mathbb{F}_p)^{N/C} \cong H^*(\mathbb{Z}/2, \mathbb{F}_2)$ equals $\mathbb{F}_2$ in all degrees. This proves the assertion for $p = 2$; so we assume p odd from now on. In this case, $H^*(\mathbb{Z}/p, \mathbb{F}_p) \cong \mathbb{F}_p[v_1, b_2]/(v_1^2, v_1 b_2 - b_2 v_1)$ with $\deg v_1 = 1$ and $\deg b_2 = 2$; see [AM, Corollary II.4.2]. Moreover, identifying $\operatorname{Aut}(\mathbb{Z}/p)$ with $\mathbb{F}_p^*$, the action of $\operatorname{Aut}(\mathbb{Z}/p)$ on $H^*(\mathbb{Z}/p, \mathbb{F}_p)$ becomes scalar multiplication, $v_1 \mapsto \ell v_1$, $b_2 \mapsto \ell b_2$, where $\ell \in \mathbb{F}_p^*$. Taking ℓ to be a generator for the subgroup of $\mathbb{F}_p^*$ corresponding to N/C, we see that

$$H^*(P, \mathbb{F}_p)^{N/C} \cong \bigoplus_{i \geq 0} \mathbb{F}_p b_2^{i[N:C]} \oplus \bigoplus_{i > 0} \mathbb{F}_p v_1 b_2^{i[N:C]-1} \ ;$$

see [AM, p. 104/105]. The smallest positive degree where $H^*(P, \mathbb{F}_p)^{N/C}$ does not vanish is therefore indeed $2([N : C] - 1) + 1 = r$.

Now assume that $\mathbb{F}_p$ is a G-module direct summand of R and R and R^G are both Cohen-Macaulay. The former hypothesis implies that $H^r(G, R) \neq 0$ and hence $\mu \leq r$. Moreover, our hypothesis on $|P|$ implies that $\mathcal{P} \ni P$ holds in Proposition 5.1, because otherwise $\mathcal{P}$ would consist of the identity subgroup alone. Therefore, height $I_R(P) \leq \mu + 1 \leq r + 1$, as desired. □

6. Multiplicative actions

6.1. In this section, we focus on a particular type of group action often called multiplicative actions. These arise from G-actions on lattices $A \cong \mathbb{Z}^n$ by extending this action k-linearly to the group algebra $R = k[A] \cong k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Here, we assume k to be a field such that $p = \operatorname{char} k$ divides the order of G; otherwise the invariant subalgebra R^G would certainly be Cohen-Macaulay because R is; see Proposition 3.1. There is no loss in assuming G to be faithfully embedded in $\mathrm{GL}(A) \cong \mathrm{GL}_n(\mathbb{Z})$, and we will do so. The above notations will remain valid throughout this section.

6.2. A subgroup $H \leq G$ acts fixed-point-freely on R if and only if no $1 \neq h \in H$ has an eigenvalue 1 on A. Furthermore, an element $g \in G$ is a bireflection on R if and only if the endomorphism $g - 1 \in \operatorname{End}(A) \cong \mathrm{M}_n(\mathbb{Z})$ has rank at most 2. Both observations are consequences of the following

Lemma 6.1. *For any subgroup $H \leq G$,* height $I_R(H) = n - \operatorname{rank} A^H$.

Proof. By definition, the ideal $I_R(H)$ of R is generated by the elements $h(a) - a = h(a)a^{-1} - 1$ for $h \in H$, $a \in A$. Thus, $R/I_R(H) \cong k[A/[H, A]]$, where we have put $[H, A] = \langle h(a)a^{-1} \mid h \in H, a \in A \rangle \leq A$. Consequently, height $I_R(H) = \dim R - \dim R/I_R(H) = n - \operatorname{rank} A/[H, A]$. Finally, since the group algebra $\mathbb{Q}[H]$ is semisimple, $A \otimes \mathbb{Q} = (A^H \otimes \mathbb{Q}) \oplus ([H, A] \otimes \mathbb{Q})$; so $\operatorname{rank} A/[H, A] = \operatorname{rank} A^H$. □

6.3. Since G permutes the k-basis A of R, the Eckmann-Shapiro Lemma implies that
$$H^*(G, R) \cong \bigoplus_{a \in G \backslash A} H^*(G_a, k) \ ,$$
where G_a denotes the isotropy group of a in G. In particular, using the notations of Sectios 5.2 and 5.5, we have
$$\mu = \inf_{a \in A} \mu_p(G_a) \ . \tag{6.1}$$

6.4. **Example: Inversion.** Let $G = \langle g = -\mathbb{I}_{n \times n} \rangle$ act on $R = k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ via $g(X_i) = X_i^{-1}$. This action is fixed-point-free. Moreover, assuming $p = 2$, we have $\mu = \mu_2(G) = 1$ by (6.1). Therefore, Corollary 5.2 gives:

R^G is Cohen-Macaulay if and only if $n \leq 2$.

6.5. **Example: Reflection groups.** An element $g \in G$ is called a *reflection* on R if height $I_R(\langle g \rangle) \leq 1$ or, equivalently, if the endomorphism $g - 1 \in \mathrm{End}(A) \cong \mathrm{M}_n(\mathbb{Z})$ has rank at most 1; see Lemma 6.1. If G is generated by reflections then R^G is an affine normal semigroup algebra over k; see [Lo$_1$]. Therefore, R^G is Cohen-Macaulay in this case, for any field k; see [BH, Theorem 6.3.5]. — This is in contrast with the situation for finite group actions on polynomial algebras by linear substitutions of the variables, where (modular) reflection groups need not lead to Cohen-Macaulay invariants [Nak].

6.6. **Cyclic Sylow subgroups.** As before, we let P denote a fixed Sylow p-subgroup of G. Moreover, $O^p(G)$ denotes the intersection of all normal subgroups N of G so that G/N is a p-group.

Theorem 6.2. *Assume that $O^p(G) \neq G$ and that P is cyclic. Then R^G is Cohen-Macaulay if and only if P is generated by a bireflection. In this case, P has order 2, 3, or 4.*

Proof. Our hypothesis $O^p(G) \neq G$ is equivalent with $\mu_p(G) = 1$; so $\mu = 1$ holds as well, by (6.1). Assuming, R^G to be Cohen-Macaulay, Corollary 5.3 and the subsequent remark imply that $G = H$. Since all p'-elements of G belong to $O^p(G)$, it follows that $G/O^p(G) = P/P \cap O^p(G)$ is generated by the images of the bireflections in P. Since P is cyclic, it follows that P is generated by a bireflection. Now, P acts faithfully on the lattice A/A^P of rank at most 2. Thus, P is isomorphic to a cyclic p-group of $\mathrm{GL}_2(\mathbb{Z})$, and these are easily seen to have orders 2, 3, or 4.

The converse follows from the more general Lemma below which does not depend on cyclicity of P or nontriviality of $G/O^p(G)$. □

Lemma 6.3. *If* $\operatorname{rank} A/A^P \leq 2$ *then R^G is Cohen-Macaulay.*

Proof. By Proposition 3.1, it suffices to show that R^P is Cohen-Macaulay; so we may assume that $G = P$ is a p-group. Note that G acts faithfully on $\overline{A} = A/A^G$. If G acts as a reflection group on $\overline{A}$ then it does so on A as well, and hence the invariants R^G will be Cohen-Macaulay; see Section 6.5. Thus we may assume that $\overline{A}$ has rank 2 and G acts on $\overline{A} \cong \mathbb{Z}^2$ as a non-reflection p-group. By the well-known classification of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ (e.g., [Lo$_2$, 2.7]), this leaves the cases $G \cong \mathbb{Z}/n$ with $n = 2, 3$ or 4 to consider.

The cases $n = 2$ or 3 can be dealt with along similar lines. Indeed, for both values of n, the only indecomposable G-lattices, up to isomorphism, are $\mathbb{Z}$, $\mathbb{Z}[G]$, and $\mathbb{Z}[G]/(\widehat{G})$, where $\widehat{G} = \sum_{g \in G} g$; see [CR, Exercise 4 on p. 514/5]. Thus, $A \cong \mathbb{Z}^m \oplus (\mathbb{Z}[G]/(\widehat{G}))^r \oplus \mathbb{Z}[G]^s$, and $R^G \cong k[B]^G[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$, where we have put $B = (\mathbb{Z}[G]/(\widehat{G}))^n \oplus \mathbb{Z}[G]^r$. Since R^G is Cohen-Macaulay if and only if $k[B]^G$ is, we may assume that $m = 0$. Now, $\overline{A} \cong (\mathbb{Z}[G]/(\widehat{G}))^{n+r}$; so $2 = (r+s)(|G|-1)$.

When $n = 3$, this leads to either $r = 1$, $s = 0$ or $r = 0$, $s = 1$. In the former case, $\operatorname{rank} A = 2$ and so R^G is surely Cohen-Macaulay, being a normal domain of dimension 2. If $r = 0$, $s = 1$ then A is a G-permutation lattice of rank 3. Hence, $R = k[A]$ is a localization of the symmetric algebra $S(A \otimes k)$, and likewise for the subalgebras of invariants. Since linear invariants of dimension ≤ 3 are known to be Cohen-Macaulay (e.g., [Ke$_2$]), R^G is Cohen-Macaulay in this case as well. For $n = 2$, there are three cases to consider, one of which ($r = 2$, $s = 0$) leads to an invariant algebra of dimension 2 which is clearly Cohen-Macaulay. Thus, we are left with the possibilities $r = 1$, $s = 1$ and $r = 0$, $s = 2$. Explicitly, after an obvious choice of basis, G acts as one of the following groups on A:

$$\textbf{Case 1: } G_1 = \left\langle g_1 = \left(\begin{array}{c|c} -1 & \\ \hline & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{array}\right) \right\rangle;$$

$$\textbf{Case 2: } G_2 = \left\langle \left(\begin{array}{c|c} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ \hline & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{array}\right) \right\rangle.$$

For G_2, $A \cong \mathbb{Z}^4$ is a permutation lattice. Hence, as above, it suffices to check that the linear invariant algebra $S(V)^G$ for $V = A \otimes k$ is Cohen-Macaulay which is indeed the case, by [ES], since $\dim V/V^G = 2$. For G_1, one can proceed as follows: Embed G_1 into $\Gamma = \langle g_1, \operatorname{diag}(-1, 1, 1)\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and denote the corresponding basis of $A \cong \mathbb{Z}^3$ by $\{x, y, z\}$; so $g_1(x) = x^{-1}$, $g_1(y) = z$, and $g_1(z) = y$. One easily checks that $R^\Gamma = k[\xi, \sigma_1, \sigma_2^{\pm 1}]$, where $\xi = x + x^{-1}$, $\sigma_1 = y + z$, and $\sigma_2 = yz$. Furthermore, $R = k[A] = R^\Gamma \oplus xR^\Gamma \oplus yR^\Gamma \oplus xyR^\Gamma$. With this, the invariant subalgebra R^{G_1} is easily determined; the result (for $\operatorname{char} k = 2$) is $R^{G_1} = R^\Gamma \oplus (xy + x^{-1}z)R^\Gamma$ which is indeed Cohen-Macaulay. This completes the proof for $G \cong \mathbb{Z}/2$ or $\cong \mathbb{Z}/3$.

We now sketch the remaining case, $G \cong \mathbb{Z}/4$. The action on $\overline{A} = A/A^G$ can then be described by $G\big|_{\overline{A}} = \langle s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\rangle$; so $\overline{A} \cong \mathbb{Z}[G]/(s^2 + 1)$. With this, one calculates $\operatorname{Ext}_G(\overline{A}, \mathbb{Z}) \cong \mathbb{Z}/2$. Thus, there is exactly one (up to isomorphism) non-split extension of G-modules $0 \to \mathbb{Z} \to U \to \overline{A} \to 0$. A suitable module U is $U = \mathbb{Z}[G]/(s - 1)(s^2 + 1)$. Furthermore, one calculates $\operatorname{Ext}_G(U, \mathbb{Z}) = 0$. Consequently, either $A \cong A^G \oplus \overline{A}$ or $A \cong \mathbb{Z}^m \oplus U$, and hence either $R^G \cong k[\overline{A}]^G[A^G]$ which is Cohen-Macaulay because $k[\overline{A}]^G$ has dimension 2, or $R^G \cong k[U]^G[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]$ which is Cohen-Macaulay precisely if $k[U]^G$ is. This reduces the problem to the case where $A = U$ which can be handled by direct calculation, taking advantage of the fact that a conjugate of group G_1 is contained in G. We leave the details to the reader. □

Acknowledgment. We thank Don Passman for the proof of Lemma 1.1 and Gregor Kemper for his comments on a preliminary version of this article.

References

[AM] A. Adem and R.J. Milgram, *Cohomology of Finite Groups*, Springer-Verlag, Berlin, 1994.

[AG] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1959), 749–765.

[Ba] A. Babakhanian, *Cohomological Methods in Group Theory*, Marcel Dekker, New York, 1972.

[Be] D.J. Benson, *Representations and Cohomology*, Vol. I, Cambridge Univ. Press, Cambridge, 1995.

[Bou] N. Bourbaki, *Algèbre commutative, chap. 5/6*, Hermann, Paris, 1964.

[BS] M.P. Brodmann and R. Y. Sharp, *Local Cohomology*, Cambridge University Press, Cambridge, 1998.

[Br] K.S. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[CHR] S.U. Chase, D.K. Harrison, and A. Rosenberg, *Galois Theory and Cohomology of Commutative Rings*, Memoirs of the Amer. Math. Soc., No. 52, Amer. Math. Soc., Providence, 1965.

[CR] C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley & Sons, New York, 1962.

[ES] G. Ellingsrud and T. Skjelbred, *Profondeur d'anneaux d'invariants en caracteristique p*, Compos. Math. **41** (1980), 233–244.

[HE] M. Hochster and J.A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[Ke_1] G. Kemper, *On the Cohen-Macaulay property of modular invariant rings*, J. Algebra **215** (1999), 330–351.

[Ke_2] G. Kemper, *Die Cohen-Macaulay Eigenschaft in der modularen Invariantentheorie*, Habilitationsschrift, Universität Heidelberg, 1999.

[Lo_1] M. Lorenz, *Multiplicative invariants and semigroup algebras*, Algebras and Representation Theory (to appear).

[Lo_2] M. Lorenz, *Class groups of multiplicative invariants*, J. Algebra **177** (1995), 242–254.

[Nak] H. Nakajima, *Invariants of finite abelian groups generated by transvections*, Tokyo J. Math. **3** (1980), 201–204.

[Ro] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, Orlando, 1979.

Department of Mathematics, Temple University, Philadelphia, PA 19122-6094
E-mail address: `lorenz@math.temple.edu`

E-mail address: `pathak@math.temple.edu`